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ABSTRACT

This activity guide is a supplement to the November 1996 issue of the National Council of Teachers of Mathematics Student Math Notes. It contains a series of problems based on one main story problem. Using this problem and the other problems contained in the guide based on it, students explore different mathematical concepts such as graph theory, counting, permutations, combinations, and Pascal's triangle. (PVD)

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How Can I Get There?

by

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from

NCTM Student Math Notes

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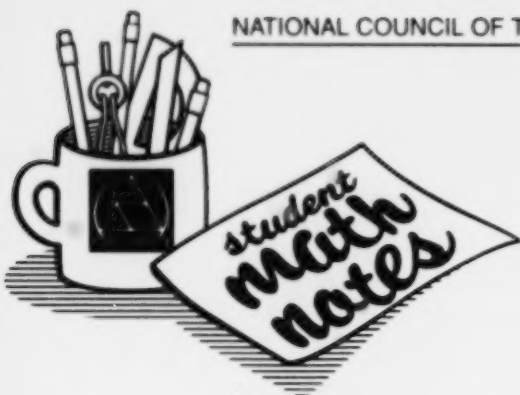
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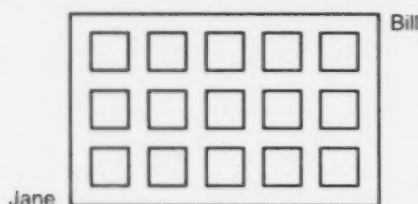
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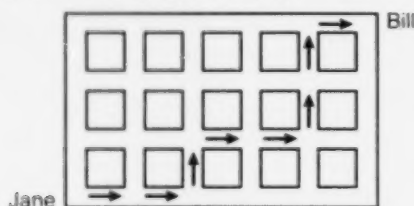
How Can I Get There?

Jane visits her buddy Bill each day. Bill lives eight blocks away, and Jane likes to vary her routine to keep her interest, since the scenery is not especially exciting. Using a map of the

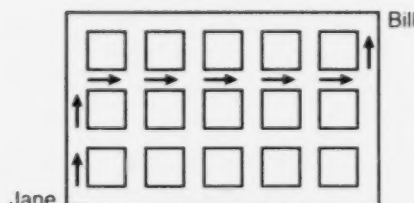
neighborhood, Jane decides to figure out how many different paths she can take if she always travels either to the right or up. She chooses to take a different eight-block route each day. She wonders how many days will pass before she has taken every possible route.



Solid arrows are used to show two possible routes. Possible route 1's can be summarized by RRURRUUR, where R is "go right" and U is "go up." Possible route 2's can be summarized by UURRRRUU.

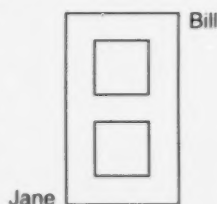


A possible route 1

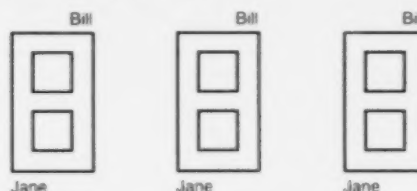


A possible route 2

Many paths are available. Jane wants to get organized so that she does not repeat any routes to Bill's house. Let us help her by first looking at the simpler example at the right.



1. Use solid arrows, one arrow for each block traveled, as shown in possible routes 1 and 2, to draw a different possible route on each diagram below. Remember, Jane has decided that she can go only either to the right or up.



2. Symbolize your three different routes using U and R as shown in the previous examples.

In our original problem, when Jane leaves her house, she must travel either to the right or up. We can indicate this path by drawing solid arrows. Thus, Jane has only one way, indicated by ① in figure 1, to get to either of the first two possible intersections by traveling either R or U.

If Jane first travels R, the two possible second blocks to travel are indicated by the clear arrows. If Jane first travels U, the two possible second blocks to travel are indicated by the dashed arrows. See figure 1.

In figure 1, Jane gets to intersection A by going RR; to get to intersection C, she must go UU. To get to intersection B, she could go either RU or UR. The total number of ways to get to B are indicated by ② in figure 2. Using U's and R's, symbolize the paths that will get Jane from her house to these locations:

3. Intersection D: _____
How many paths are possible? _____
4. Intersection E: _____
How many paths are possible? _____
5. Intersection F: _____
How many paths are possible? _____
6. Intersection G: _____
How many paths are possible? _____

How Can I Get There?—Continued

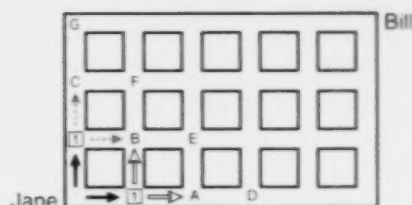


Fig. 1

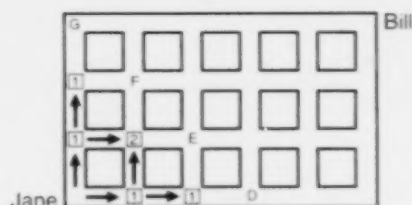


Fig. 2

Because Jane can go only to the right or up, the only way she can get to intersection D is from intersection A. Since only one way exists to get to intersection A, only one way exists to get to intersection D. Jane can get to intersection E from either intersection A or intersection B. Since only one way exists to get to intersection A and two ways exist to get to intersection B, $2 + 1$, or 3, ways are available to get to intersection E.

7. From which two intersections can Jane get to intersection F? _____
8. How many paths can she take to get to intersection F? _____
9. From which intersections can Jane get to intersection G? _____
10. How many paths can she take to get to intersection G? _____
11. Look for a pattern to find the number of paths to get to each intersection along the route from Jane's house to Bill's house. Record these numbers at the appropriate intersections in **figure 3**.

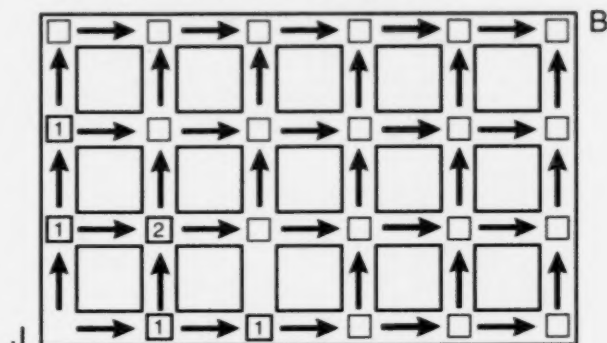


Fig. 3

12. Because Jane will travel a different path each day, how many days will pass before she has taken every possible route? _____

13. If each possible path is symbolized with U's and R's, how many U's are in a path? _____ R's? _____

You should have determined from problem 13 that each of Jane's routes can be represented using a string of 5 R's and 3 U's in some order. The answer found in question 12 is the number of different ways to write 5 R's and 3 U's in a string. Each string is called a *permutation* of those eight letters. We can use this idea to help Jane when Bill moves.

In the situation shown in **figure 4**, Jane must travel four blocks to get to Bill. Let us look at this trip from a different perspective by considering the number of ways to arrange four different letters.

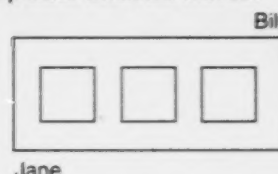


Fig. 4

14. List all the arrangements of A, B, C, and D by completing the following chart.

ABCD	BACD	CABD	DABC
ABDC	B		
ACBD	B		
ACDB	B		
ADBC	B		
ADCB	B		

How does this process relate to finding the number of ways to arrange the letters of the string URRR? Suppose that the string of letters is U rR. Those letters can be arranged in the following ways with the letter U first: U rR, U Rr, Ur R, UR r, and URr. Since each R looks different from the others, these six arrangements do not look the same. The R's in our example are the same, so these arrangements can all be represented by URRR.

15. Of the twenty-four arrangements of the letters in the string ABCD, how many choices are available for the first letter? _____
16. Once that first letter is chosen, how many choices are possible for the second letter? _____
17. Once the first two letters are chosen, how many choices remain for the third letter? _____

After the first three letters are in place, only one letter is left to put into the fourth spot. Thus, the total number of arrangements can be represented by the product $4 \cdot 3 \cdot 2 \cdot 1$. This result is the same as the one found in the chart in problem 14. A mathematical symbol called "4 factorial" represents this product:

$$4! = 4 \cdot 3 \cdot 2 \cdot 1$$

How Can I Get There?—Continued

18. Represent the number of arrangements of the letters in the string RRR using factorial notation. _____

Since each R in the arrangement RRR is the same, only one arrangement is possible: RRR . Thus, the number of arrangements of $URRR$ is different from, but related to, the number of arrangements of $ABCD$. If four letters are different, $4!$ arrangements of them exist. If three of the four letters are identical, the number of arrangements is reduced by a factor of $3!$. Thus,

$$\frac{4!}{3!} = 4$$

ways to arrange the letters in the string $URRR$ are possible. Those four paths are shown in **figure 5**. They correspond to $URRR$ (①), $RURR$ (②), $RRUR$ (③), and $RRRU$ (④).

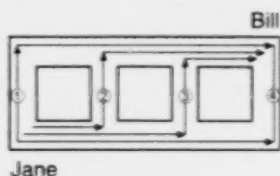


Fig. 5

Consider a different situation, shown in **figure 6**.

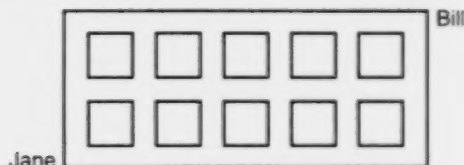


Fig. 6

19. How many blocks must Jane travel? _____
How many to the right? _____ Up? _____

This process is like finding the number of distinct arrangements of the string $RRRRRUU$.

20. How many arrangements of seven different letters are possible? _____
Of five different letters? _____
Of two different letters? _____

If we follow a similar procedure for $RRRRRUU$, we would find $7!$ arrangements of the seven letters, of which $5!$ are repeats of the R 's and $2!$ are repeats of the U 's. As a result,

$$\frac{7!}{5! \cdot 2!} = 21$$

distinct arrangements of $RRRRRUU$ are possible. We can conclude that Jane has twenty-one different routes to follow to get to Bill's house.

We can use permutations to find the number of distinct paths in a different way. Remember that in the original problem, Jane had eight blocks to travel, five of which were to the right and three were up. Using these procedures for arranging letters, we see that Jane can take

$$\frac{8!}{5! \cdot 3!} = 56$$

distinct routes.

Looking back, we find that Jane can get to her final destination from one of two places: intersection K or intersection L (**fig. 7**). Both places are seven blocks from Jane's house. Intersection K requires her traveling five blocks to the right and two blocks up, as shown in question 19. Intersection L requires traveling four blocks to the right and three blocks up. By the reasoning used previously, we know that Jane can get to intersection K in

$$\frac{7!}{5! \cdot 2!} = 21$$

ways and to intersection L in

$$\frac{7!}{4! \cdot 3!} = 35$$

ways. Adding, we find that Jane can get to Bill's house in $21 + 35 = 56$ distinct routes.

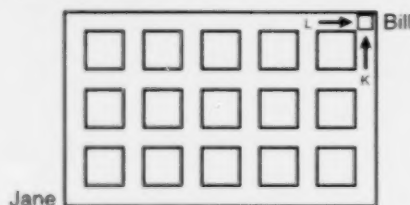


Fig. 7

Symbolizing this process, we find that

$$\frac{8!}{5! \cdot 3!}$$

can be written

$$\frac{7!}{5! \cdot 2!} + \frac{7!}{4! \cdot 3!}$$

21. What do you notice in comparing the numerators of

$$\frac{7!}{5! \cdot 2!} \text{ and } \frac{7!}{4! \cdot 3!}$$

with the numerator of

$$\frac{8!}{5! \cdot 3!} ?$$

22. What do you notice in comparing the denominators of

$$\frac{7!}{5! \cdot 2!} \text{ and } \frac{7!}{4! \cdot 3!}$$

with the denominator of

$$\frac{8!}{5! \cdot 3!} ?$$

How Can I Get There?—Continued

23. Use the observations in questions 21 and 22 to answer the following question. If

$$\frac{10!}{6! \cdot 4!}$$

represents the number of different routes for a complete trip, what two fractions represent the number of ways to get to the two places from which the final block can be traveled? _____ and _____

24. Verify that the sum of these two fractions is

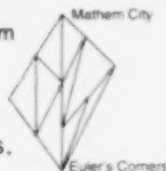
$$\frac{10!}{6! \cdot 4!}$$

Did you know that ...

- Jane's original problem is analogous to moving from the point (0, 0) to the point (5, 3) on the coordinate grid if moves are restricted to one unit to the right or one unit up?
- this concept can be extended to three, four, ..., n dimensions?
- this concept can be extended to finding the paths between two cities?
- this activity relates to the binomial expansion of $(R + U)^n$?

Can you ...

- find the number of distinct ways to move from the point (-4, 2) to the point (5, -3) by moving either to the right or down?
- extend this procedure to count the ways a king can move on a chessboard? Remember that a king can also move on a diagonal.
- determine the number of paths for Jane to travel to Bill's house if she must walk around a park that covers two adjacent blocks?
- find the number of distinct minimal paths to travel, on edges of the unit cubes, from one corner point to the opposite corner of a rectangular solid 5 units by 4 units by 3 units built from unit cubes?
- find the number of paths traveling south from Mathem City to Euler's Corners?



Mathematical Content

Graph theory, counting, permutations, combinations, Pascal's triangle

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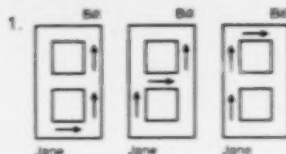
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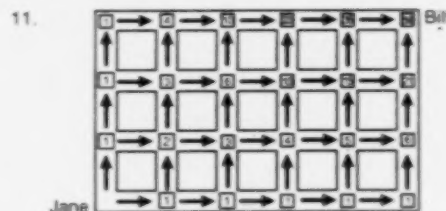
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Answers:



1. RUU, URU, UUR
3. RRR, 1
4. RRU, RUR, URR, 3
5. RUU, URU, UUR, 3
6. UUU, 1
7. B and C
8. $1 + 2 = 3$
9. C
10. 1



11. 56 days

13. 3, 5

14.

ABCD	BACD	CABD	DABC
ABDC	BADC	CADB	DACB
ACBD	BCAD	CBAD	DBAC
ACDB	BCDA	CBDA	DBCA
ADBC	BDAC	CDAB	DCAB
ADCB	BDCA	CDBA	DCBA

15. 4

16. 3

17. 2

18. $3!$

19. 7, 5, 2

20. $7!, 5!, 2!$

21. In each addend, the number before the factorial symbol is 1 less than the number before the factorial symbol in the sum.

22. In one addend, the first number before the factorial symbol is 1 less than the first number before the factorial symbol in the sum; in the other addend, the second number before the factorial symbol is 1 less than the second number before the factorial symbol in the sum.

$$23. \frac{9!}{6!3!} + \frac{9!}{5!4!}$$

$$24. \frac{9!}{6!3!} + \frac{9!}{5!4!} = \frac{4}{4} \cdot \frac{9!}{6!3!} + \frac{6}{6} \cdot \frac{9!}{5!4!} = \frac{4 \cdot 9! + 6 \cdot 9!}{6!4!} = \frac{10 \cdot 9!}{6!4!} = \frac{10!}{6!4!}$$